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Applications of exact solutions to the Navier–Stokes equations: free shear layers

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A family of exact solutions to the Navier–Stokes equations is used to analyse unsteady three-dimensional viscometric flows that occur in the vicinity of a plane boundary that translates and rotates with time-varying velocities. Such flows are important in the study of flows that are produced by rotating machinery. They are also useful in describing local behaviour in more complex global flows, such as that produced in a shear layer by the passage of a disturbance in the mainstream. An example is the flow produced in a turbulent shear layer by the passage of the core of a Rankine vortex. When the effect of viscosity is unimportant, the use of Lagrangian coordinates reduces the mathematical problem to that of solving a set of linear ordinary differential equations.

1. Introduction

This study uses a family of exact solutions to the Navier–Stokes equations governing unsteady flows of a viscous incompressible fluid to analyse several technically important flows. Generally, if (x_1, x_2, x_3) denote the Cartesian coordinates of a point in the fluid, these solutions are characterized by the fact that the expressions for the velocity components (u_1, u_2, u_3) are linear forms in x_1 with coefficients that are functions of (t, x_2, x_3). The reference axes may translate and rotate relative to an inertial frame with velocities that vary with t. Most of the well-known solutions of the Navier–Stokes equations belong to this family. Our primary aim is to show that there are many other solutions of the same form, and that these solutions can be used to analyse real flows. Moreover, these solutions can be generalized to study flows of viscoelastic fluids.

This first paper describes a sub-class of such flows, designated $F1$ flows. These satisfy the additional requirement that $u_3 = u_3(t, x_3)$, so that the component of fluid velocity in the direction normal to the plane $x_3 = 0$ is a function only of time, t, and the distance, x_3, from that plane. The plane can rotate and translate with time-varying velocities relative to an inertial frame. In addition to being exact solutions to the Navier–Stokes equations, $F1$ solutions are exact solutions to the boundary-layer equations: the terms that are usually omitted to obtain these approximate equations from the Navier–Stokes equations are identically zero. Well-known examples of parallel $F1$ flows are Couette (1890), Rayleigh (1911) and Ekman (1905) flows. Examples of non-parallel $F1$ flows are the various Von Kármán (1921) swirling flows, and the Blasius–Hiemenz (1908,1911) flows at a forward stagnation point. In these examples, the fluid is either contained between two parallel plane surfaces, or occupies a semi-infinite region bounded by a single plane surface. For Couette and Rayleigh flows the bounding surfaces translate in directions parallel to their planes, but do not rotate; for Ekman flows the bounding surfaces rotate about the same fixed axis with the same constant total angular velocity and translate in directions parallel to their planes. Von Kármán flows are generated when two parallel plane surfaces rotate with different angular speeds about a common fixed axis that is normal to their planes. The Blasius–Hiemenz flow approximates that produced near a line of symmetry when a fluid is squeezed between two parallel plates moving with different speeds normal to their planes.

All of the viscometric-type flows listed above are produced when the motions of the bounding plane surfaces have at most two degrees of freedom. The $F1$ solutions to the Navier–Stokes equations presented here may be used to describe flows that are produced when the bounding surfaces move in *any* manner that is compatible with them remaining plane rigid and parallel: they can translate in directions parallel and normal to their planes and rotate about different axes. The mathematical problem reduces to that of solving a system of nonlinear partial differential equations in the two independent variables (t, x_3). This can be split into two sets: the first set, which is usually nonlinear, describes the primary component of the flow and is identical to that governing one of the flows listed above; it uncouples from the second set which, essentially, is linear with coefficients that are determined by the primary component of the flow.

Two sub-classes of $F1$ flows are of special interest because they involve relatively simple mathematics. The simpler one, which is described here, is characterized by the fact that the primary component of the flow is such that the coefficients in the second set of linear equations are functions only of t. These can be transformed into two, uncoupled, constant coefficient diffusion equations. This result was used by Kambe & Tsutomu (1983) and Kambe (1986) to discuss the interaction of two parallel shear layers as the distance between them decreases. For the second sub-class of $F1$ flows the primary component of the flow is steady, and is governed by a set of nonlinear ordinary differential equations; the second set of linear partial differential equations has coefficients that are functions only of x_3. Techniques for integrating such equations are described by Varley & Seymour (1988). Examples of these flows have been given previously by Rott & Lewellen (1967) who described the boundary layer above a flat plate that rotates and translates, by Abbott & Walters (1970) who described the flow between two flat plates that rotate with the same angular speed about different axes, and by Smith (1987) who described a class of rotating eccentric flows.

Although their most obvious application is to the study of viscometric flows and to flows that are produced by rotating machinery, in this first paper we show how $F1$ solutions may be used also to analyse some features of the strong local interaction between a thin shear layer, in which the flow is initially parallel, and a disturbance in the outer mainstream. An excellent example of such a problem occurs in the study of the strong *local* atmospheric disturbance that is produced in a layer of air, where the flow is strongly sheared by the passage of a tornado. Here, the lower boundary of the layer, representing the Earth, is fixed and the upper boundary, representing the cloud base, is rotating and translating. Mainly, we concentrate on the flow in regions away from material boundaries where it is the fact that the fluid is strongly sheared that controls the interaction with the mainstream flow and not the presence of rigid boundaries. In this sense the layer is a free shear layer, and the governing equations can be transformed into constant-coefficient diffusion equations.

The general problem is formulated in § 3. It is supposed that prior to $t = 0$ the flow is an unsteady parallel shear flow in which $u_3 \equiv 0$. This ambient flow could be any one of the well-known parallel $F1$ flows or, as we show in § 3.2, it could be a turbulent shear flow that, like the Rayleigh and Ekman flows, is modelled by similarity solutions to the equations governing $F1$ flows. After $t = 0$, the passage of a disturbance in the mainstream subjects the fluid in the shear layer to non-uniform time-varying pressure gradients. Typically, as we show in § 4, this mainstream flow can be described by solutions to the Euler equations. When these Euler flows are also $F1$ flows, the governing equations are most easily integrated by using Lagrangian coordinates. In fact, using these variables reduces the problem of solving a system of nonlinear partial differential equations to that of solving a system of linear ordinary differential equations. Examples of mainstream flows are given in § 5.

In § 6 we describe the flow produced at the edge of a shear layer during the passage of the core of a Rankine vortex in the mainstream. Far from the shear layer, in which the flow may be turbulent, the axis of the vortex is perpendicular to the layer and moves with a time-varying velocity; the strength of the vortex also varies with time. This flow models that occurring in the centre of a tornado which, far above the Earth, is convected with the ambient wind speed. The resulting flow is unsteady, and fully three-dimensional.

For most of the flows discussed in this paper the axis of rotation of the reference frame is fixed relative to an inertial frame. In § 9 and § 8 we state the modifications that should be made to the analyses when this axis also rotates with a time-varying angular velocity. The representations are used to describe flows in a pitching channel.

The exact solutions to the Navier–Stokes equations that are used in this study describe highly idealized flows: the boundaries of the flow regions are usually of infinite extent. In spite of this, like the classical Rayleigh and Von Kármán solutions, sometimes these solutions can be used to provide useful information about local conditions in more complicated, and more realistic, global flows. For example, $F1$ flows can be used to approximate real flows in the vicinity of an axis of rotation or a line of symmetry. It should be noted, however, that these are *similarity* solutions of the Navier–Stokes equations that, like all other similarity solutions, must be interpreted with care: they can be used only when it can be argued that the local flow is well approximated by a viscometric flow.

2. Formulation

The flows are referred to Cartesian axes that are either inertial or are *translating* and *rotating* relative to inertial axes with velocities that vary in time, t. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ denote the Cartesian coordinates of a point in the flow at which the fluid velocity is $\boldsymbol{u}(t, \boldsymbol{x})$. Then, using conventional vector notation, the Navier–Stokes equations governing unsteady motions of an incompressible fluid are

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega}' \times \boldsymbol{x} + \nabla p = \nu \nabla^2 \boldsymbol{u}. \tag{2.1}$$

$\boldsymbol{\Omega}(t)$ is the angular velocity of the reference axes with respect to inertial axes;

$$p(t, \boldsymbol{x}) = \overline{p}/\rho + \boldsymbol{f} \cdot \boldsymbol{x} + \frac{1}{2}((\boldsymbol{\Omega} \cdot \boldsymbol{x})^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})(\boldsymbol{x} \cdot \boldsymbol{x})), \tag{2.2}$$

where $\overline{p}$ is the pressure, ρ is the constant density of the fluid, and $\boldsymbol{f}(t)$ is the acceleration of the origin relative to an inertial frame. If the fluid is subject to a conservative body force $-\rho\nabla\chi$ per unit volume, the term $\chi(t,\boldsymbol{x})$ should be added to the expression for p. Also, with an obvious re-interpretation of $\boldsymbol{\Omega}(t)$ and p, (2.1) may be used to model the effect of a magnetic field whose strength and direction vary with time.

In this study, exact solutions to the Navier–Stokes equations are used to analyse several technically important flows for which $\boldsymbol{u}$ and p take the forms

$$\boldsymbol{u} = \boldsymbol{w}(t,x_2,x_3)x_1 + \boldsymbol{v}(t,x_2,x_3) \tag{2.3}$$

and

$$p = \tfrac{1}{2}P_2(t,x_2,x_3)x_1^2 + P_1(t,x_2,x_3)x_1 + P_0(t,x_2,x_3). \tag{2.4}$$

2.1. *F1 flows*

These are viscometric flows with velocity and pressure fields that are special cases of those given by (2.2)–(2.4). They are characterized by the condition that the component of fluid velocity, v, in the direction normal to some plane, which may translate and rotate relative to inertial axes with time-varying velocities, is a function only of t and the distance, y, from that plane. If the reference axes are chosen so that $x_3 = y$, for $F1$ flows

$$u_i = a_{ij}(t,y)x_j + b_i(t,y), \quad i,j = 1,2, \quad u_3 = v(t,y), \tag{2.5}$$

and

$$p = \tfrac{1}{2}k_{ij}(t)x_ix_j + p_i(t,y)x_i + p_0(t,y) \quad \text{with} \quad k_{ij} = k_{ji}, \quad i,j = 1,2. \tag{2.6}$$

The representations (2.5) and (2.6) are special cases of those given by (2.3) and (2.4) with

$$\left.\begin{aligned} &w_i = a_{i1}(t,x_3), \quad v_i = a_{i2}(t,x_3)x_2 + b_i(t,x_3), \quad i = 1,2, \quad w_3 = 0, \quad v_3 = v(t,x_3), \\ &P_2 = k_{11}(t), \quad P_1 = k_{12}(t)x_2 + p_1(t,x_3), \quad P_0 = \tfrac{1}{2}k_{22}(t)x_2^2 + p_2(t,x_3)x_2 + p_0(t,x_3). \end{aligned}\right\} \tag{2.7}$$

The expressions (2.5) for the velocity components also satisfy the equations governing the boundary-layer flow adjacent to the surface $y = 0$: the terms that are usually omitted to obtain the boundary layer-equations from the Navier–Stokes equations are identically zero. In the mainstream the pressure gradients parallel to the surface $y = 0$ are

$$\frac{\partial p}{\partial x_i} = k_{ij}(t)x_j + p_i(t,y), \quad i,j = 1,2. \tag{2.8}$$

Mainly, we consider flows for which the x_3-axis also coincides with the axis of rotation of the reference frame. Accordingly,

$$\boldsymbol{\Omega} = (0,0,\Omega) \quad \text{and} \quad p = \bar{p}/\rho + \boldsymbol{f}\cdot\boldsymbol{x} - \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2). \tag{2.9}$$

When the expressions (2.5) and (2.6) are inserted, the Navier–Stokes equations (2.1) imply that $v(t,y)$ and $\mathbf{a}(t,y) = (a_{ij})$, $i,j = 1,2$, satisfy the equations

$$\left.\begin{aligned} &\frac{\partial v}{\partial y} + a_{11} + a_{22} = 0, \\ \text{and} \quad &\frac{\partial \mathbf{a}}{\partial t} + v\frac{\partial \mathbf{a}}{\partial y} + (\mathbf{a} + 2\Omega\boldsymbol{J})\mathbf{a} + \Omega'\boldsymbol{J} + \mathbf{k} = \nu\frac{\partial^2 \mathbf{a}}{\partial y^2}, \end{aligned}\right\} \tag{2.10}$$

where $\boldsymbol{J} = (-\epsilon_{ij3})$, $i,j = 1,2$. ($J_{11} = J_{22} = 0, -J_{12} = J_{21} = 1$.) Once $v(t,y)$ and $\mathbf{a}(t,y)$ have been determined from (2.10) and the associated auxiliary conditions, $\boldsymbol{b}(t,y) = (b_1,b_2)$ is determined from the equation

$$\frac{\partial \boldsymbol{b}}{\partial t} + v\frac{\partial \boldsymbol{b}}{\partial y} + (\mathbf{a} + 2\Omega\boldsymbol{J})\boldsymbol{b} + \boldsymbol{p}(t) = \nu\frac{\partial^2 \boldsymbol{b}}{\partial y^2}, \tag{2.11}$$

together with associated auxiliary conditions. Finally, $p_0(t,y)$ is determined from $v(t,y)$ by the condition

$$\frac{\partial p_0}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial t} - v\frac{\partial v}{\partial y}. \tag{2.12}$$

$F1$ flows are used to model the behaviour of global flows in the vicinity of the x_3-axis. In (2.10) and (2.11), $\boldsymbol{k}(t)$ and $\boldsymbol{p}(t)$ are arbitrary: in practice, these functions are chosen either to model the effect of the global flow on that near the x_3-axis, or as eigenfunctions so that boundary conditions that are imposed at selected values of x_3 at this span are satisfied. Couette (1890), Rayleigh (1911), Ekman (1905), Blasius–Hiemenz (1908,1911) stagnation point, and von Kármán (1921) swirling flows have velocity and pressure fields of the forms (2.5) and (2.6).

3. Shear-layer flows

In this paper $F1$ solutions to the Navier–Stokes equations are used to describe the flow produced in a localized shear layer by the passage of a disturbance in the mainstream. The shear layer contains, or is adjacent to, the surface $y = 0$, which may or may not be a material surface. Prior to the arrival of the disturbance the flow inside the shear layer is parallel to the plane $y = 0$, and is described by those solutions to (2.10) and (2.11) for which $v \equiv 0$. The mainstream flow, which may also be sheared in the y-direction, is described by exact solutions to the inviscid forms of these equations.

3.1. *Parallel flows*

We consider parallel flows that are described by solutions to (2.10) and (2.11) of the form

$$v \equiv 0, \quad \boldsymbol{a} = (\Omega_0 - \Omega(t))\boldsymbol{J}; \quad \text{and} \quad \boldsymbol{k} = (\Omega_0^2 - \Omega^2(t))\boldsymbol{I}_0, \tag{3.1}$$

where Ω_0 is a constant and $\boldsymbol{I}_0$ is the unit (identity) matrix. For these flows, the y-component of vorticity is $\omega(t)=(a_{21}\text{-}a_{12})/2$ and the y-component of total vorticity

$$\omega(t) + \Omega(t) \equiv \text{constant } = \Omega_0. \tag{3.2}$$

The two-dimensional velocity vector

$$\boldsymbol{u} = (u_1, u_2) = (\Omega_0 - \Omega)\boldsymbol{J}\boldsymbol{x} + \boldsymbol{b}(t,y), \tag{3.3}$$

where, according to (2.10), $\boldsymbol{b}(t,y) = (b_1, b_2)$ satisfies

$$\frac{\partial \boldsymbol{b}}{\partial t} + (\Omega_0 + \Omega)\boldsymbol{J}\boldsymbol{b} + \boldsymbol{p} = \nu\frac{\partial^2 \boldsymbol{b}}{\partial y^2}. \tag{3.4}$$

In particular, when $\Omega \equiv$ constant, $= \Omega_0$, $\boldsymbol{u} = (b_1, b_2)$ where

$$\frac{\partial b_1}{\partial t} - 2\Omega_0 b_2 + p_1 = \nu\frac{\partial^2 b_1}{\partial y^2} \quad \text{and} \quad \frac{\partial b_2}{\partial t} + 2\Omega_0 b_1 + p_2 = \nu\frac{\partial^2 b_2}{\partial y^2}. \tag{3.5}$$

These equations govern Couette, Rayleigh and Ekman flows.

All solutions to (3.4) can be written as

$$\boldsymbol{b} = \boldsymbol{b}_0(t) + [\boldsymbol{I}(t)]^{-1}\boldsymbol{C}(t,y), \tag{3.6}$$

where $\boldsymbol{C}(t,y)$ satisfies the linear diffusion equation

$$\frac{\partial \boldsymbol{C}}{\partial t} = \nu\frac{\partial^2 \boldsymbol{C}}{\partial y^2}, \tag{3.7}$$

$$\boldsymbol{l}(t) = \begin{pmatrix} \cos\gamma(t) & \sin\gamma(t) \\ -\sin\gamma(t) & \cos\gamma(t) \end{pmatrix} \quad \text{with} \quad \gamma(t) = \int_0^t (\Omega_0 + \Omega(t'))\mathrm{d}t', \tag{3.8}$$

and
$$\boldsymbol{b}_0(t) = \boldsymbol{l}^{-1}[\boldsymbol{b}_0(0) - \int_0^t \boldsymbol{l}(t')\boldsymbol{p}(t')\mathrm{d}t']. \tag{3.9}$$

We list the forms of $\boldsymbol{b}_0(t)$ and $\boldsymbol{C}(t,y)$ corresponding to four technically important *self-similar* parallel shear flows. For each of these flows $\boldsymbol{a}(t) \equiv 0$ and $\boldsymbol{u} = \boldsymbol{b}(t,y)$. For the first two $\Omega = \Omega_0 = 0$ and, according to (3.8) and (3.9),

$$\boldsymbol{l}(t) \equiv \boldsymbol{l}_0 \quad \text{and} \quad \boldsymbol{b}_0(t) = \boldsymbol{b}_0(0) - \int_0^t \boldsymbol{p}(t')\mathrm{d}t'. \tag{3.10}$$

The first example is the unsteady Stokes–Rayleigh flow that is produced above a plate $y=0$ when the fluid is started impulsively from rest with velocity $\boldsymbol{U}_e$ at $t=-T$. Then,

$$\boldsymbol{b}_0 \equiv \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{C}(t,y) = \boldsymbol{U}_e \mathrm{erf}(y/2(\nu(T+t))^{1/2}). \tag{3.11}$$

The second example is the unsteady flow that is produced in an unbounded fluid by the diffusion of a shear layer that was centred on the plane $y=0$ at $t=-T$. Then,

$$\boldsymbol{b}_0 \equiv 0 \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{C}(t,y) = \boldsymbol{M}(4\pi\nu(t+T))^{-1/2}\exp(-y^2/4\nu(t+T)). \tag{3.12}$$

The constant

$$\boldsymbol{M} = \int_{-\infty}^{\infty} \boldsymbol{b}(t,y)\mathrm{d}y \tag{3.13}$$

measures the strength of the shear layer.

For the third example, an Ekman flow, the reference axes are rotating with constant angular velocity Ω about the y-axis and, according to (3.8),

$$\boldsymbol{l}(t) = \begin{pmatrix} \cos(2\Omega t) & \sin(2\Omega t) \\ -\sin(2\Omega t) & \cos(2\Omega t) \end{pmatrix}. \tag{3.14}$$

When $\boldsymbol{p} \equiv$ constant, for the *steady* flow above the stationary plane $y=0$

$$\boldsymbol{b} = (2\Omega)^{-1}[\boldsymbol{l}_0 - \exp(-(\Omega/\nu)^{1/2}y)\boldsymbol{l}(-y/2(\nu\Omega)^{1/2})]\boldsymbol{J}\boldsymbol{p}, \tag{3.15}$$

$$\boldsymbol{b}_0 = (2\Omega)^{-1}\boldsymbol{J}\boldsymbol{p}, \tag{3.16}$$

and

$$\boldsymbol{C} = -(2\Omega)^{-1}\exp(-(\Omega/\nu)^{1/2}y)\boldsymbol{J}(t - y/2(\nu\Omega)^{1/2})]\boldsymbol{J}\boldsymbol{p}. \tag{3.17}$$

Equation (3.17) implies that

$$C_1 + \mathrm{i}C_2 = (2\mathrm{i}\Omega)^{-1}(p_1 + \mathrm{i}p_2)\exp((-2\mathrm{i}\Omega/\nu)^{1/2}y)\exp(-2\mathrm{i}\Omega t). \tag{3.18}$$

3.2. *Self-similar flows: turbulent shear layers*

The expressions (3.11) and (3.12) for each component of $\mathbf{C}$ are examples of solutions to the diffusion equation that can be written in the similarity form

$$C = U_1(1+t/T)^{-r/2}S(\eta,r), \quad \text{where} \quad \eta = y/(2\nu(T+t))^{1/2} \tag{3.19}$$

and $S(\eta,r)$ satisfies the Rayleigh equation

$$S'' + \eta S' + rS = 0. \tag{3.20}$$

In (3.19) and (3.20), U_1, T and r are constants and the $'$ denotes differentiation with respect to η. Actually, the expression (3.18) for $(C_1 + \mathrm{i}C_2)$ may be obtained also as the limit of an expression of the form (3.19) by taking

$$T = -\mathrm{i}(r/4\Omega), \quad U_1 = (2\mathrm{i}\Omega)^{-1}(p_1 + \mathrm{i}p_2) \tag{3.21}$$

and then letting $r \to \infty$ at constant (t, y). In this limit

$$(1 + t/T)^{-r/2} \to \exp(-2\mathrm{i}\Omega t) \quad \text{and} \quad S \to \exp((-2\mathrm{i}\Omega/\nu)^{1/2} y). \tag{3.22}$$

Since the components of **C** satisfy the linear diffusion equation, more complicated shear flows can be obtained by replacing the one-term expression for $C(t, y)$ given by (3.19) by a sum of such expressions. For example, if $S(\eta, r)$ denotes the function satisfying (3.20) together with the conditions

$$S(0, r) = 0 \quad \text{and} \quad \eta^r S(\eta, r) \to 1 \quad \text{as} \quad \eta \to \infty, \tag{3.23}$$

and if

$$\eta_0 = y/(2\nu(T_0 + t))^{1/2}, \tag{3.24}$$

then

$$C = U_0 S(\eta_0, 0) + U_1[S(\eta, 0) - (1 + t/T)^{-r/2} S(\eta, r)]/r \tag{3.25}$$

is a *composite* similarity solution to (3.7). According to (3.23) and (3.25), if $U = (\nu/2T)^{1/2}$,

$$\text{as } (\eta, \eta_0) \to \infty, \; C \sim U_0 + U_1[1 - (Uy/\nu)^{-r}]/r. \tag{3.26}$$

When $r > 0$, in the mainstream

$$C = U_0 + r^{-1} U_1, = U_e. \tag{3.27}$$

As $r \to 0$ the asymptotic profile (3.26) is often used to *curve-fit* the shear profile that occurs in the outer part of a turbulent boundary layer. If U denotes the 'friction velocity', typically

$$U_0 \approx 5.5U \;\text{ and }\; U_1 \approx 2.5U. \tag{3.28}$$

When $r = 0$ relation (3.26) implies that at large, but *finite*, values of (Uy/ν),

$$C/U \sim 5.5 + 2.5 \ln(Uy/\nu). \tag{3.29}$$

In this limit

$$C = U_0 S(\eta_0, 0) + U_1[\ln(1 + t/T)^{1/2} S(\eta, 0) + S_1(\eta)], \tag{3.30}$$

where $S_1(\eta)$ satisfies the equation

$$S_1'' + \eta S_1' = S(\eta, 0) \quad \text{with} \quad S_1(0) = 0 \quad \text{and} \quad S_1(\eta)/\ln(\eta) \to 1 \quad \text{as} \quad \eta \to \infty. \tag{3.31}$$

It can be shown that when $r < 2$,

$$S(\eta, r) = (2/\pi)^{1/2} \exp(-\eta^2/2) \int_0^\infty \sinh(\eta z) \exp(-z^2/2) z^{-r} \mathrm{d}z, \tag{3.32}$$

which implies that $S'(0, r) = 2^{(1-r)/2}\Gamma(1 - r/2)/\pi^{\frac{1}{2}}$. In particular, $S(\eta, 0) = \mathrm{erf}(\eta/\sqrt{2})$. Also,

$$S_1(\eta) = (2/\pi)^{1/2} \exp(-\eta^2/2) \int_0^\infty \sinh(\eta z) \exp(-z^2/2) \ln(z) \mathrm{d}z. \tag{3.33}$$

When $t = 0$,

$$\eta = (Uy/\nu) \quad \text{and} \quad \eta_0 = (T/T_0)^{1/2}(Uy/\nu). \tag{3.34}$$

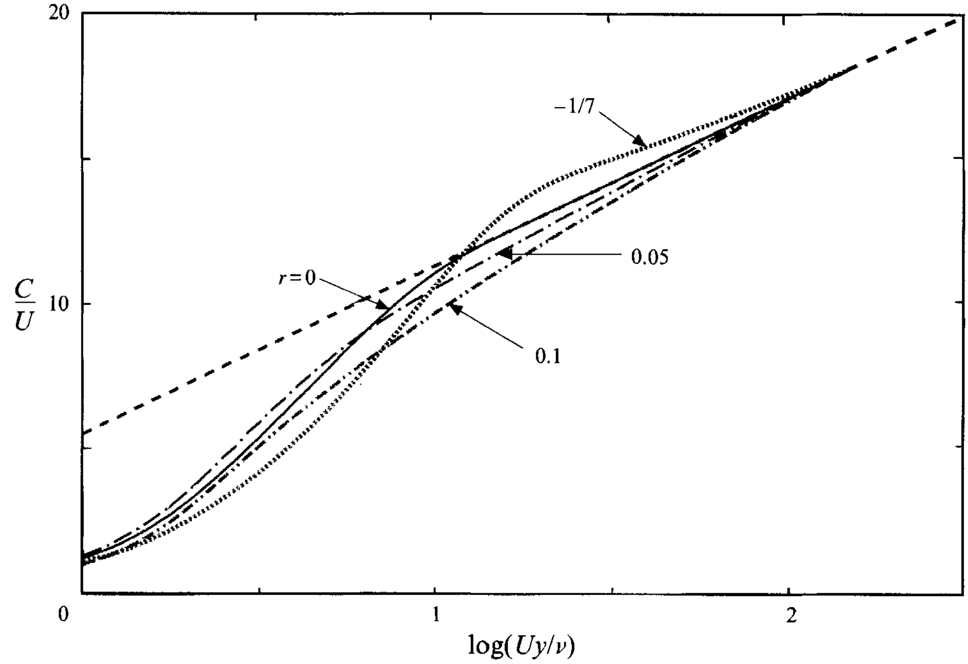

FIGURE 1. Profiles of $C(0, y)$ given by (3.30) for various values of r. The dashed curve depicts the log-profile given by equation (3.29).

Figure 1 shows the profile of $C(0, y)$ given by (3.30) for *all* y in the range where $0 \leqslant Uy/\nu < 10^2$ when U_0 and U_1 are given by (3.28) and when $T/T_0 \approx 0.04$, the value that ensures

$$U = (\nu C_y(0,0))^{1/2}. \tag{3.35}$$

The profile is in good agreement with that found experimentally during the period when the shear profile near the wall is changing slowly on the timescale T. For comparison, figure 1 also shows $C(0, y)$ for other values of r. The parameters U_0/U, U_1/U and (T/T_0) are chosen so that (3.35) is satisfied and the asymptotic law (3.29) is approximated. When $U_0 + r^{-1}U_1 = 0$, condition (3.26) implies that away from the wall $C \sim U_0(Uy/\nu)^{-r}$. This power-law profile, with $r \approx -1/7$, is also used to curve-fit turbulent shear profiles.

The use of analytic expressions of the form (3.25) to model turbulent flows during the 'quiescent' stage has been discussed in detail by Walker *et al.* (1989).

4. Inviscid mainstream flows: Lagrangian formulation

We suppose that prior to $t = 0$ the flow is a parallel shear flow of the type described in § 3. Thereafter, the shear layer is disturbed by the passage of a disturbance in the mainstream that subjects the fluid in the shear layer to non-uniform time-varying pressure gradients. In the mainstream conditions are changing slowly with distance in all directions parallel to the plane $y = 0$ and, relative to an observer moving with the disturbance, the pressure gradients can be approximated locally by expressions of the form (2.8).

The flow in the mainstream is governed by the Euler equations. In the special cases when the $\partial p/\partial x_i$, $i = 1, 2$, are *known* functions of (t, x_1, x_2), these equations are most easily integrated by introducing Lagrangian variables. In fact, *when the $\partial p/\partial x_i$ are given by (2.8), with $p_i = p_i(t)$, using Lagrangian variables reduces the problem of solving the inviscid forms of (2.10) and (2.11), which are a set of nonlinear partial differential equations, to that of solving a system of linear ordinary differential equations.* For, if $(\boldsymbol{x}(t, \boldsymbol{X}, Y), y(t, \boldsymbol{X}, Y))$ denote the coordinates of the particle that had coordinates $(\boldsymbol{X}, Y)$ at $t = 0$, the Euler equations imply that

$$\boldsymbol{x}'' + \boldsymbol{J}(2\Omega \boldsymbol{x}' + \Omega' \boldsymbol{x}) + \nabla p = 0, \tag{4.1}$$

where $\nabla p = (\partial p/\partial x_1, \partial p/\partial x_2)$ and $'$ denotes the derivative with respect to t at constant $(\boldsymbol{X}, Y)$. This second-order equation must be solved subject to the initial conditions that

$$\text{when} \quad t = 0, \quad \boldsymbol{x} = \boldsymbol{X} \quad \text{and} \quad \boldsymbol{x}' = \boldsymbol{u}(0, \boldsymbol{X}, Y), = \boldsymbol{U}(\boldsymbol{X}, Y). \tag{4.2}$$

Then, once $\boldsymbol{x}(t, \boldsymbol{X}, Y)$ has been found, $y(t, \boldsymbol{X}, Y)$ is determined from the equation

$$\frac{\partial(\boldsymbol{x}, y)}{\partial(\boldsymbol{X}, Y)} = 1 \tag{4.3}$$

and the condition that

$$\text{when } Y = 0, \quad y = y_0(t, \boldsymbol{x}); \tag{4.4}$$

$y = y_0(t, \boldsymbol{x})$ is the equation of the bounding material surface $Y = 0$. In terms of $\boldsymbol{x}(t, \boldsymbol{X}, Y)$ and $y(t, \boldsymbol{X}, Y)$

$$\boldsymbol{u} = \boldsymbol{x}' \quad \text{and} \quad v = y'. \tag{4.5}$$

Van Dommelen (1981) was the first to use Lagrangian variables effectively to study *non-interactive* flow problems for which $p(t, \boldsymbol{x})$ can be estimated *a priori.* Blythe, Kazakia & Varley (1972), Varley, Kazakia & Blythe (1977), and Varley & Blythe (1983) used $(t, \mathbf{x}, Y)$ as independent variables to study several *interactive* flow problems for which the determination of $p(t, \mathbf{x})$ was part of the problem.

When the $\partial p/\partial x_i$ are given by (2.8) with $p_i = p_i(t)$, (4.1) reduces to the *linear* equation

$$\boldsymbol{x}'' + 2\Omega \boldsymbol{J}\boldsymbol{x}' + (\boldsymbol{k} + \Omega' \boldsymbol{J})\boldsymbol{x} + \boldsymbol{p} = 0. \tag{4.6}$$

Also, when the velocity field is of the form (2.5), the initial conditions (4.2) require that

$$\text{when } t = 0, \quad \boldsymbol{x} = \boldsymbol{X} \quad \text{and} \quad \boldsymbol{x}' = \boldsymbol{A}(Y)\boldsymbol{X} + \boldsymbol{B}(Y), \tag{4.7}$$

where

$$\boldsymbol{A}(Y) = \boldsymbol{a}(0, Y) \quad \text{and} \quad \boldsymbol{B}(Y) = \boldsymbol{b}(0, Y) \tag{4.8}$$

are specified functions. The solution to (4.6) satisfying conditions (4.7) can be written as

$$\boldsymbol{x} = \boldsymbol{g}_0(t)\boldsymbol{X} + \boldsymbol{g}_1(t)\boldsymbol{U} + \boldsymbol{x}_0(t), \tag{4.9}$$

where $\boldsymbol{g}_0(t)$ and $\boldsymbol{g}_1(t)$ satisfy the homogeneous form of (4.6), ($\boldsymbol{p} = 0$), with

$$\boldsymbol{g}_0(0) = \boldsymbol{I}_0, \quad \boldsymbol{g}_0'(0) = \boldsymbol{0}, \quad \boldsymbol{g}_1(0) = \boldsymbol{0}, \quad \boldsymbol{g}_1'(0) = \boldsymbol{I}_0, \tag{4.10}$$

and $\mathbf{x}_0(t)$ satisfies (4.6) with

$$\boldsymbol{x}_0(0) = \boldsymbol{x}_0'(0) = \boldsymbol{0}. \tag{4.11}$$

Also,

$$\boldsymbol{U} = \boldsymbol{A}(Y)\boldsymbol{X} + \boldsymbol{B}(Y). \tag{4.12}$$

It follows that

$$\boldsymbol{x} = \boldsymbol{g}(t, Y)\boldsymbol{X} + \boldsymbol{h}(t, Y), \tag{4.13}$$

where

$$\boldsymbol{g}(t, Y) = \boldsymbol{g}_0(t) + \boldsymbol{g}_1(t)A(Y) \quad \text{and} \quad \boldsymbol{h}(t, Y) = \boldsymbol{x}_0(t) + \boldsymbol{g}_1(t)\boldsymbol{B}(Y). \tag{4.14}$$

According to (4.13),

$$\boldsymbol{u} = \boldsymbol{x}' = \boldsymbol{g}'X + \boldsymbol{h}' = \boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}, \tag{4.15}$$

where

$$\boldsymbol{a} = \boldsymbol{g}'\boldsymbol{g}^{-1} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{h}' - \boldsymbol{a}\boldsymbol{h}. \tag{4.16}$$

The representations obtained above for $\boldsymbol{a}(t, Y)$ and $\boldsymbol{b}(t, Y)$ may be obtained directly from the inviscid forms of equations (2.10) and (2.11). These imply that $\boldsymbol{a}(t, Y)$ and $\boldsymbol{b}(t, Y)$ satisfy the Riccati-type equations

$$\boldsymbol{a}' + (\boldsymbol{a} + 2\Omega \boldsymbol{J})\boldsymbol{a} + \Omega' \boldsymbol{J} + \boldsymbol{k} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b}' + (\boldsymbol{a} + 2\Omega \boldsymbol{J})\boldsymbol{b} + \boldsymbol{p} = \boldsymbol{0}. \tag{4.17}$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are written in terms of $\boldsymbol{g}$ and $\boldsymbol{h}$ as in (4.16), it follows that $\boldsymbol{g}(t, Y)$ and $\boldsymbol{h}(t, Y)$ satisfy linear equations whose solutions can be written in the form (4.14)

Relations (4.15) and (4.16) determine $\boldsymbol{u}$ as an explicit function of $(t, \boldsymbol{x}, Y)$. Then $y(t, Y)$ is determined from condition (4.3) which implies that

$$\frac{\partial y}{\partial Y} = \beta^{-1}, \quad \text{where} \quad \beta = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \det(\boldsymbol{g}) \tag{4.18}$$

is a function only of (t, Y). If $y = y_0(t)$ is the equation of the material surface $Y = 0$, according to (4.18)

$$y = y_0(t) + \int_0^Y \frac{\mathrm{d}S}{\beta(t, S)} \quad \text{and} \quad v = \frac{\partial y}{\partial t}. \tag{4.19}$$

The velocity components determined from relations (4.6)–(4.19) represent *exact* solutions to the Euler equations.

5. Mainstream flows for which $\boldsymbol{a} = \boldsymbol{a}(t)$.

We consider mainstream flows for which $\boldsymbol{a} = \boldsymbol{a}(t)$, where $\boldsymbol{a}(t)$ is given by (3.1) for $t < 0$ and is determined in terms of $\boldsymbol{k}(t)$ from the first of conditions (4.17) for $t > 0$. The second of these conditions then implies that $\boldsymbol{b}(t, y)$ may still be written in the form given by (3.6) and (3.9). $\boldsymbol{l}(t)$ is determined from the conditions

$$\boldsymbol{l}' = \boldsymbol{l}(\boldsymbol{a} + 2\Omega \boldsymbol{J}) \quad \text{and} \quad \boldsymbol{l}(0) = \boldsymbol{l}_0, \tag{5.1}$$

while

$$\boldsymbol{C} = \boldsymbol{B}(Y) - \boldsymbol{b}_0(0) \quad \text{with} \quad Y = \beta(t)(y - y_0(t)); \tag{5.2}$$

$\beta(t)$ satisfies the equation

$$\beta' = (a_{11} + a_{22})\beta \quad \text{with} \quad \beta(0) = 1. \tag{5.3}$$

Consequently, in the mainstream

$$\boldsymbol{u} = \boldsymbol{a}(t)\boldsymbol{x} + \boldsymbol{b}_0(t) + [\boldsymbol{l}(t)]^{-1}(\boldsymbol{B}(Y) - \boldsymbol{b}_0(0)) \tag{5.4}$$

and

$$v = -\frac{\beta'}{\beta}(y - y_0) + y_0'. \tag{5.5}$$

The motions of the vortex lines are of some interest. When $\mathbf{u}$ is given by (5.4) the vorticity vector

$$(\omega_1, \omega_2, \omega) = \frac{1}{2}\left(-\frac{\partial b_2}{\partial y}, \frac{\partial b_1}{\partial y}, a_{21} - a_{12}\right). \tag{5.6}$$

It follows that the equation of the vortex line through the point $(\boldsymbol{x}, y) = (\boldsymbol{x}_0, y_0)$ is

$$\boldsymbol{x} = \boldsymbol{x}_0 + [2\omega(t)]^{-1}\boldsymbol{J}(\boldsymbol{b}(t, y) - \boldsymbol{b}(t, y_0)). \tag{5.7}$$

5.1. *Plane flows*

As a first example of a mainstream disturbance, consider the case when the reference axes are not rotating, when $\mathbf{u} = (u, 0)$, and when the flow variables are independent of x_2. Accordingly, in the mainstream, with $x = x_1$,

$$\left.\begin{aligned} p &= \tfrac{1}{2}k(t)x^2 + p_1(t)x + \tfrac{1}{2}Q_2(t)(y - y_0)^2 + Q_1(t)(y - y_0), \\ u &= a(t)x + b_0(t) + (B(Y) - b_0(0))/\beta(t), \\ \text{and} \quad v &= y_0'(t) - a(t)(y - y_0(t)); \end{aligned}\right\} \tag{5.8}$$

$a(t)$ and $b_0(t)$ are related to $k(t)$ and $p_1(t)$ by the equations

$$a' + a^2 + k = 0 \quad \text{and} \quad b_0' + ab_0 + p_1 = 0. \tag{5.9}$$

In terms of $\beta(t)$,

$$a = \beta'/\beta, \quad b_0 = \beta^{-1}[b_0(0) - \int_0^t \beta(t')p_1(t')\mathrm{d}t'], \tag{5.10}$$

and

$$\boldsymbol{I} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.11}$$

According to (5.9) and (5.10), $\beta(t)$ satisfies the equation

$$\beta'' + k\beta = 0 \quad \text{with} \quad \beta(0) = 1 \quad \text{and} \quad \beta'(0) = a(0). \tag{5.12}$$

Also, it follows from (2.12) and (5.8) that

$$Q_1(t) = -y_0'' \quad \text{and} \quad Q_2(t) = a' - a^2. \tag{5.13}$$

If

$$\overline{x} = x - x_R(t), \quad \overline{y} = y - y_0(t) \quad \text{and} \quad \overline{v} = v - y_0(t), \tag{5.14}$$

where

$$x_R(t) = \left(\int_0^t \beta(t')p_1(t')\mathrm{d}t'\right)/\beta'(t), \tag{5.15}$$

relations (5.8) can be re-written

$$u = [\beta'(t)\overline{x} + B(\beta(t)\overline{y})]/\beta(t) \quad \text{and} \quad \overline{v} = -\frac{\beta'(t)}{\beta(t)}\overline{y}. \tag{5.16}$$

The velocity field given by (5.14)–(5.16) is an exact solution of the Euler equations.

According to (5.16), in the mainstream the $(\overline{x}, \overline{y})$ coordinates of the particle that had (Lagrangian) coordinates (X, Y) at $t = 0$ are given by

$$\overline{x} = \overline{x}_0(t) + \beta(t)X + \beta_1(t)B(Y) \quad \text{and} \quad \overline{y} = Y/\beta(t), \tag{5.17}$$

where

$$\overline{x}_0(t) = -\beta(t)\int_0^t [\beta(\tau)]^{-1}x_R'(\tau)\mathrm{d}\tau \quad \text{and} \quad \beta_1(t) = \beta(t)\int_0^t [\beta(\tau)]^{-2}\mathrm{d}\tau. \tag{5.18}$$

Also, the streamfunction

$$\psi = [\beta'(t)\overline{x}Y + \int_0^Y B(Y')\mathrm{d}Y']/\beta^2(t) \quad \text{with} \quad Y = \beta(t)\overline{y}. \tag{5.19}$$

As an application of the representations (5.8)–(5.19), consider the flow above the stationary flat plate $y = 0$ when

$$B(0) = 0, \quad B(\infty) = U_e, \quad B'(Y) \geqslant 0, \quad \text{and} \quad k(t) \geqslant 0. \tag{5.20}$$

These conditions ensure that the pressure gradient in the mainstream is increasing with increasing x, which is the direction of the basic flow at $t = 0$, and that $\beta'(t)/\beta(t) \leqslant 0$ for some interval $0 \leqslant t \leqslant t_c$. It then follows from (5.8) that

$$\text{as } y \to 0, \quad u \to \beta'(t)(x - x_R(t))/\beta(t), = u_e(t, x); \tag{5.21}$$

$u_e > 0$ when $x < x_R(t)$ and $u_e < 0$ when $x > x_R(t)$. Thus, according to inviscid theory, viewed relative to the plate the flow downstream of the cross-section $x = x_R(t)$ contains a region of reverse flow. When $k(0) \neq 0$,

$$x_R(0) = -p_1(0)/k(0), \tag{5.22}$$

which, according to (5.10), is the value of x where $\partial p/\partial x = 0$. For small times,

$$x_R(t) \sim x_R(0) - \tfrac{1}{2}p_1'(0)t/k(0), \tag{5.23}$$

while at the cross-section where $\partial p/\partial x = 0$

$$x \sim x_R(0) - p_1'(0)t/k(0). \tag{5.24}$$

The inviscid flow described by relations (5.16) is self-similar with similarity variables

$$Y = \beta(t)y \quad \text{and} \quad \xi = \beta'(t)(x - x_R(t)). \tag{5.25}$$

In terms of these variables, the equation of the instantaneous streamline through the point $(\xi, Y) = (\xi_0, Y_0)$ is

$$\xi = \left(\int_{Y_0}^{Y} B(S)\mathrm{d}S + \xi_0 Y_0\right) / Y. \tag{5.26}$$

Also, the equation of the critical level where $u = 0$ is

$$\xi = B(Y), \tag{5.27}$$

while the equation of the dividing streamline is

$$\xi = \left(\int_0^Y B(S)\mathrm{d}S\right) / Y. \tag{5.28}$$

Figure 2 shows a typical streamline pattern predicted by inviscid theory when

$$B(Y) = U_e \operatorname{erf}(Y/Y_0). \tag{5.29}$$

This profile is depicted by the broken curve which, according to (5.27), also represents the critical level. According to (5.28) the shear layer separates from the plate when $\xi = U_e$ which, together with (5.25), implies that

$$x = x_R(t) - U_e/\beta'(t) = x_S(t). \tag{5.30}$$

With some minor modifications, the above analysis may be used to describe some features of the early stages of the breakup of a free shear layer due to the passage of a disturbance in the outer flow. Figure 3 depicts a typical streamline pattern when

$$B(Y) = M(\pi Y_0)^{-1/2} \exp(-(Y/Y_0)^2). \tag{5.31}$$

The adverse pressure gradients induced by the passage of the disturbance outside the

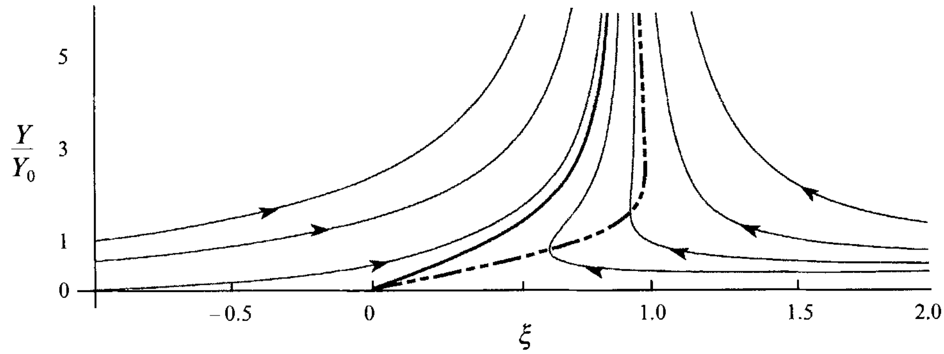

FIGURE 2. The streamline pattern in the neighbourhood of the separation point in the self-similar flow when $B(Y)$ is given by equation (5.29). The thicker solid and broken curves represent the dividing streamline and the critical level.

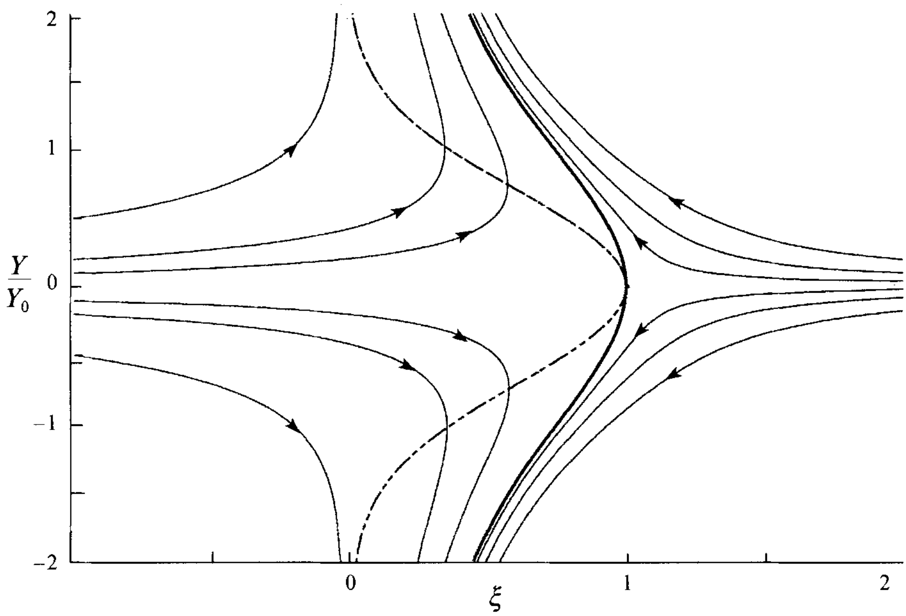

FIGURE 3. The streamline pattern in the neighbourhood of the curve where an unsteady flow stagnates. $B(Y)$ is given by equation (5.31). The solid and broken curves represent the dividing streamline and the critical level.

shear layer cause the flow to stagnate at the broken curve. The flow divides at the heavy solid curve.

5.2. *Turbulent mainstream*

When $y_0=0$ and

$$B(Y)=U_0+r^{-1}U_1(1-(Uy/\nu)^{-r}), \tag{5.32}$$

relations (5.16) imply that

$$\left.\begin{aligned} u &= [\beta'(t)(x-x_R(t))+U_0+r^{-1}U_1(1-(\beta(t)Uy/\nu)^{-r})]/\beta(t) \\ \text{and}\qquad v &= -[\beta'(t)/\beta(t)]y. \end{aligned}\right\} \tag{5.33}$$

In (5.33), $\beta(t)$ is determined by the time variation of $\partial^2 p/\partial x^2 = k(t)$ in the mainstream from (5.12). For most of the problems discussed in this study the mainstream disturbance is specified, and $\beta(t)$ is determined uniquely. If, however, $\partial^2 p/\partial x^2$ fluctuates in time about some average value, as it does if the flow in the mainstream is turbulent, the simple flow defined by relations (5.33) may also be used to model some features of that found in the *outer* part of a turbulent boundary layer. The dependence of u on x is required if the flow is to converge or diverge. At any fixed point, the turbulence in the mainstream may be intermittent: for times when the convergence or divergence is strong the first term in the expression for u dominates, and the vorticity is weak; when the convergence or divergence is weak, u is approximated by the well-established power-law profile in a turbulent flow. When $r = 0$, (5.33) implies

$$u = [\beta'(t)(x - x_R(t)) + U_0 + U_1 \ln(\beta(t)Uy/\nu)]/\beta(t). \tag{5.34}$$

5.3. *Rankine vortex*

As a second example of a mainstream disturbance, consider the case when the disturbance is produced by the passage of the core of a Rankine vortex whose axis of rotation is parallel to the y-axis. If $x = x_1 + \mathrm{i}x_2$ and $u = u_1 + \mathrm{i}u_2$, for this flow

$$p = k(t)(x_1^2 + x_2^2)/2 + p_1(t)x_1 + p_2(t)x_2 + Q_2(t)y^2/2 + Q_1(t)y, \tag{5.35}$$

$$u = (\delta(t) + i\omega(t))x + b_0(t) \quad \text{and} \quad v = v_0(t) - 2\delta(t)y. \tag{5.36}$$

Conditions (4.17) imply that $k(t), \delta(t)$ and $\omega(t)$ are related by the condition

$$(\delta + \mathrm{i}\omega)' + (\delta + \mathrm{i}\omega)^2 + k = 0, \tag{5.37}$$

and that $b_0(t)$ satisfies the equation

$$b_0{}' + (\delta + \mathrm{i}\omega)b_0 + p_1 + \mathrm{i}p_2 = 0. \tag{5.38}$$

$Q_1(t)$ and $Q_2(t)$ can be expressed in terms of $v_0(t)$ and $\delta(t)$ as

$$Q_1 = -(v_0' - 2\delta v_0) \quad \text{and} \quad Q_2 = -(2\delta' - 4\delta^2). \tag{5.39}$$

Relation (5.37) implies that we can write

$$(\delta + \mathrm{i}\omega) = \phi'/\phi \quad \text{where} \quad \phi'' + k\phi = 0 \quad \text{with} \quad \phi(0) = 1. \tag{5.40}$$

Also, according to (5.38) and (5.40),

$$b_0(t) = \phi^{-1}\left[b_0(0) - \int_0^t \phi(t')(p_1(t') + \mathrm{i}p_2(t'))\mathrm{d}t'\right]. \tag{5.41}$$

Conditions (5.40), together with the fact that k is real, imply that if the variation in strength of the vortex core, $\omega(t)$, is specified

$$\phi(t) = [\beta(t)]^{1/2}\exp(\mathrm{i}\Phi(t)) \quad \text{where} \quad \beta(t) = \omega(0)/\omega(t) \quad \text{and} \quad \Phi(t) = \int_0^t \omega(t')\mathrm{d}t'. \tag{5.42}$$

Also,

$$\delta(t) = -\omega'(t)/2\omega(t), \quad \text{and} \quad I(t) = [\beta(t)]^{\frac{1}{2}}\begin{pmatrix} \cos\Phi(t) & -\sin\Phi(t) \\ \sin\Phi(t) & \cos\Phi(t) \end{pmatrix}; \tag{5.43}$$

$k(t)$ can be expressed in terms of $\omega(t)$ by using the first of conditions (5.43) and the fact, which follows from (5.37), that

$$k(t) = \omega^2(t) - \delta^2(t) - \delta'(t). \tag{5.44}$$

If the reference axes are rotating about the y-axis with an angular velocity $\Omega(t)$ relative to an inertial frame, ω should be replaced by $\omega + \Omega$ and k in (5.37) should be replaced by $k + \Omega^2$.

The trajectory of the axis, or centre, of rotation is given by

$$x = -b_0/(\delta + \mathrm{i}\omega) = c(t). \tag{5.45}$$

If the flow is referred to a polar coordinate system based on this axis,

$$u_r = \delta(t)r \quad \text{and} \quad u_\theta = \omega(t)\mathrm{r}. \tag{5.46}$$

The equation of the instantaneous streamline through the point (x_0, y_0) can be written in the form

$$\frac{x-c}{x_0-c} = \left(\frac{y_0-y_s}{y-y_s}\right)^\kappa, \tag{5.47}$$

where

$$y_s = v_0/2\delta \quad \text{and} \quad \kappa = (\delta + \mathrm{i}\omega)/2\delta. \tag{5.48}$$

If the swirling flow in the mainstream is also sheared in the y-direction

$$u = (\delta(t) + \mathrm{i}\omega(t))(x - c(t)) + B(Y)/\phi(t), \tag{5.49}$$

where

$$Y = \omega(0)y/\omega(t) \quad \text{and} \quad B(Y) = B_1(Y) + \mathrm{i}B_2(Y). \tag{5.50}$$

$\phi(t)$ and $\delta(t)$ are determined from $\omega(t)$ by (5.42) and (5.43), $c(t)$ is given by (5.45), and v is given by (5.36). When $c = y_s = 0$, the equations of the instantaneous streamline and vortex line through the point (x_0, y_0) are

$$x = x_0(y_0/y)^\kappa - (2\delta\phi)^{-1}Y^{-\kappa}\int_{Y_0}^{Y} B(S)S^{\kappa-1}\mathrm{d}S, \quad \text{where} \quad Y_0 = \omega(0)y_0/\omega(t), \tag{5.51}$$

and

$$x = x_0 + (\tfrac{1}{2}\mathrm{i})(\omega_0\omega)^{-1/2}[B(Y) - B(Y_0)]\mathrm{e}^{-\mathrm{i}\Phi}. \tag{5.52}$$

The equation for the particle trajectories is

$$x = (\omega_0\omega)^{-1/2}[\omega_0 X\mathrm{e}^{\mathrm{i}\Phi} + B(Y)\sin(\Phi)], \quad (\omega_0 = \omega(0)). \tag{5.53}$$

Figure 4 shows how the streamsurfaces change with increasing ω/ω_0. These figures correspond to δ = constant, so that, according to (5.42) and (5.43),

$$\omega = \omega_0\exp(-2\delta t), \quad \Phi = (\omega_0 - \omega)/2\delta, \quad \text{and} \quad \phi = (\omega_0/\omega)^{1/2}\mathrm{e}^{(\Phi)}. \tag{5.54}$$

Also, $B(Y)$ is given by (5.29); $\omega/\omega_0 = 1$ for figure 4(a), and $\omega/\omega_0 = 2$ for figure figure 4(b).

6. Free shear layers

The mainstream flows described in § 5 were assumed to be governed by the Euler equations. However, when $\boldsymbol{a} = \boldsymbol{a}(t)$ the corresponding solutions to the Navier–Stokes equations can easily be found. The viscous term in (2.10) is zero, while equation (2.11) implies that $\boldsymbol{b}(t, Y)$ satisfies the equation

$$\frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{a} + 2\Omega\boldsymbol{J})\boldsymbol{b} + \boldsymbol{p} = \nu\beta^2\frac{\partial^2 \boldsymbol{b}}{\partial Y^2}. \tag{6.1}$$

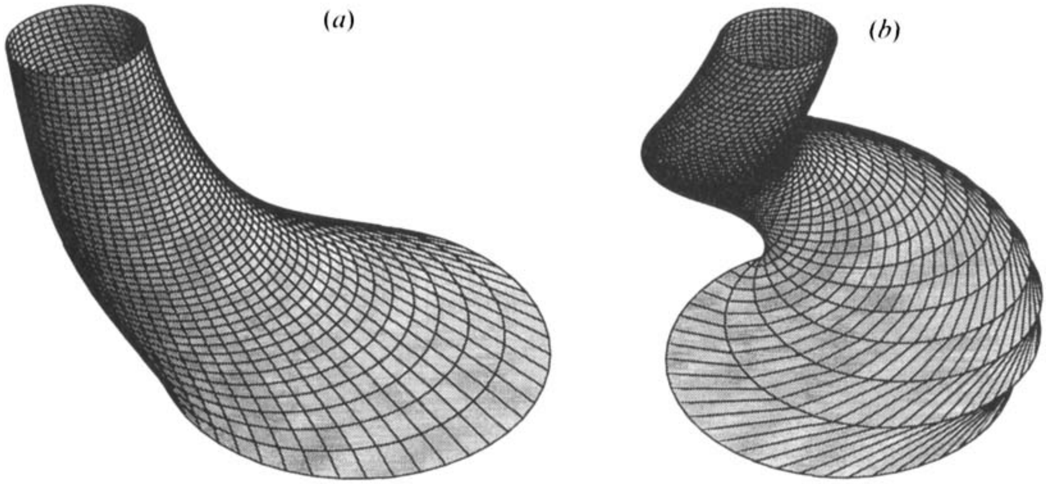

FIGURE 4. A streamsurface in a sheared flow that is disturbed by the passage of the core of a Rankine vortex: (a) $\omega/\omega_0 = 1$; (b) $\omega/\omega_0 = 2$.

In terms of the time measure

$$\tau = \int_0^t \beta^2(t')\mathrm{d}t', \tag{6.2}$$

the solution to (6.1) can be written

$$\boldsymbol{b} = \boldsymbol{b}_0(t) + [\boldsymbol{l}(t)]^{-1}\boldsymbol{C}(\tau, Y), \tag{6.3}$$

where $\mathbf{C}(\tau, Y)$ satisfies the linear diffusion equation (3.7). The representation (6.3) is of the same form as that given by (3.6).

To summarize: when $\boldsymbol{a} = \boldsymbol{a}(t)$ satisfies the first of (4.17) for some symmetric matrix $\boldsymbol{k}(t)$, the velocity fields are of the form

$$\boldsymbol{u} = \boldsymbol{a}(t)\boldsymbol{x} + \boldsymbol{b}_0(t) + [\boldsymbol{l}(t)]^{-1}\boldsymbol{C}(\tau, Y) \quad \text{and} \quad v = -\frac{\beta'}{\beta}(y - y_0) + y_0'; \tag{6.4}$$

$\boldsymbol{b}_0(t)$ is given by (3.9), $\boldsymbol{l}(t)$ is determined from (5.1), and Y is given by (5.2) with $\beta(t)$ determined from (5.3). $C_1(\tau, t)$ and $C_2(\tau, Y)$, the components of $\boldsymbol{C}(\tau, Y)$, are *any* functions that satisfy the *linear constant-coefficient* diffusion equation. Parallel shear flows that satisfy the additional conditions (3.1) are special cases of those described by relations (6.4). For these flows $\beta(t) \equiv 1$, $\tau = t$ and $Y = y$.

One possible representation for $\mathbf{C}(\tau, Y)$ is

$$\boldsymbol{C} = \mathrm{Re}\left[\sum_\kappa \overline{\boldsymbol{C}}_\kappa \exp(\mathrm{i}\kappa Y - \nu\kappa^2\tau)\right], \quad \text{or} \quad \boldsymbol{C} = \mathrm{Re}\left[\int \overline{\boldsymbol{C}}(\kappa)\exp(\mathrm{i}\kappa Y - \nu\kappa^2\tau)\mathrm{d}\kappa\right], \tag{6.5}$$

where the summation is over a discrete range of the complex parameter κ, and the integration is over some continuous range of variation of κ. The fact that the Navier–Stokes equations have *exact* solutions for which the velocity components can be written in the form (6.4) with $\mathbf{C}(\tau, Y)$ given by (6.5) was known by Kelvin (1887) (see Craik & Criminale 1986.) When the flow domain is the region where $Y > 0$, for $\tau > 0$ a more useful representation for $\mathbf{C}(\tau, Y)$ is

$$\mathbf{C}(\tau, Y) = \int_0^\infty [G(\tau, Y - Y') - G(\tau, Y + Y')]\mathbf{C}(0, Y')dY' + \int_0^\tau G_1(\tau - \tau', Y)\mathbf{C}(\tau', 0)d\tau', \tag{6.6}$$

where

$$G(\tau, Y) = (4\pi\nu\tau)^{-1/2}\exp[-Y^2/(4\nu\tau)] \quad \text{and} \quad G_1(\tau, Y) = -2\nu\frac{\partial g}{\partial Y}. \tag{6.7}$$

In (6.6),

$$\boldsymbol{C}(0, y) = \boldsymbol{b}(0, y) - \boldsymbol{b}_0(0) \quad \text{and} \quad \boldsymbol{C}(\tau, 0) = l(t)(\boldsymbol{b}(t, y_0) - \boldsymbol{b}_0(t)), \tag{6.8}$$

where $\tau(t)$ is given by (6.2). If the flow domain is the infinite region where $-\infty < Y < \infty$, for $\tau > 0$

$$\boldsymbol{C}(\tau, Y) = \int_{-\infty}^\infty G(\tau, Y - Y')\boldsymbol{C}(0, Y')\mathrm{d}Y'. \tag{6.9}$$

In the inviscid limit, the first integral in (6.6) and the integral in (6.9) $\to \mathbf{C}(0, Y)$ and, for all $Y > 0$, the second integral in (6.6)$\to \mathbf{0}$.

This study describes some features of the interaction of a localized shear layer with a disturbance in the mainstream. When this layer is adjacent to a plane rigid surface, $Y = 0$, at which no-slip conditions must be applied, we must take $\boldsymbol{a} = \boldsymbol{a}(t, y)$ in (2.5). Velocity fields of the form (6.4) cannot satisfy such boundary conditions. They may be used to describe the interaction only for times when the presence of boundaries has no appreciable effect on the flow: thereafter $\boldsymbol{a} = \boldsymbol{a}(t, y)$. For such early times the shear layer may be regarded as a free shear layer in which $\boldsymbol{a} = \boldsymbol{a}(t)$ and the contribution of the second integral in (6.6) is negligible compared with that of the first: the flow is determined by the shear profile at $t = 0$ and by the disturbance in the mainstream. Accordingly, either $\boldsymbol{C}(\tau, Y)$ is given by (6.9) or

$$\boldsymbol{C}(\tau, Y) = \int_0^\infty [G(\tau, Y - Y') - G(\tau, Y + Y')]\boldsymbol{C}(0, Y')\mathrm{d}Y'. \tag{6.10}$$

When the flow prior to $t = 0$ is one of the four parallel flows discussed in § 3, the appropriate integral (6.9) or (6.10) can be evaluated explicitly: $\boldsymbol{C}(\tau, Y)$ in (6.4) is obtained by simply replacing (t, y) by (τ, Y) in the expression that is used for $\boldsymbol{C}(t, y)$ in the parallel flow. For example, when the parallel flow is a plane flow in which $\boldsymbol{u} = (u, 0)$, where $u = C(t, y)$ is given by (3.25), and if the mainstream disturbance is also two-dimensional, for $t > 0$

$$\left.\begin{aligned} &u = \beta^{-1}(\beta'(x - x_R) + U_0\mathrm{erf}(\eta_0/\sqrt{2}) + r^{-1}U_1[\mathrm{erf}(\eta/\sqrt{2}) - (1 + \tau/T)^{-r/2}S(\eta, r)]) \\ &\text{and} \\ &\qquad v = -(\beta'/\beta)y, \\ &\text{where} \quad \eta = \beta y/(2\nu(T + \tau))^{1/2} \quad \text{and} \quad \eta_0 = \beta y/(2\nu(T_0 + \tau))^{1/2}; \end{aligned}\right\} \tag{6.11}$$

$\beta(t)$ and $x_R(t)$ are given by (5.12) and (5.15) while $\tau(t)$ is given by (6.2). The velocity field given by (6.11) is an exact solution to the Navier–Stokes equations.

Alternatively, if the velocity field in the parallel flow is given by

$$u = u_1 + \mathrm{i}u_2 = C_1(t,y) + \mathrm{i}C_2(t,y) = C(t,y), \tag{6.12}$$

and if the mainstream disturbance is that produced by the passage of the core of a Rankine vortex, the representations (6.4) imply that

$$u = (\delta(t) + \mathrm{i}\omega(t))(x - c(t)) + [\phi(t)]^{-1}C(\tau, Y), \tag{6.13}$$

where $\phi(t), \delta(t)$, and $c(t)$ are determined in terms of $\omega(t)$ by (5.42)–(5.45), Y is given by (5.50), and

$$\tau = \omega^2(0)\int_0^t \frac{\mathrm{d}s}{\omega^2(s)}; \tag{6.14}$$

v is given by (5.36). When $c = y_s = 0$, the equations of the instantaneous streamline and vortex line through the point (x_0, y_0) are

$$x = x_0(y_0/y)^\kappa - (2\delta\phi)^{-1}y^{-\kappa}\int_{y_0}^{y} C(\tau, \beta s)s^{\kappa-1}\mathrm{d}s \tag{6.15}$$

and

$$x = x_0 + \tfrac{1}{2}\mathrm{i}[\omega(0)\omega(t)]^{-1/2}\exp[-\mathrm{i}\Phi(t)][C(\tau(t), Y) - C(\tau(t), Y_0)]; \tag{6.16}$$

β, Y and Y_0 are given by (5.42), (5.50) and (5.51). The trajectory of the particle that had coordinates (X, Y) at $t = 0$ is determined from the relations

$$x = \phi(t)[X + \int_0^t [\phi(t')]^{-2}C(\tau(t'), Y)\mathrm{d}t'], \quad y = \omega(t)Y/\omega(0). \tag{6.17}$$

When δ = constant,

$$\tau = ((\omega_0/\omega)^2 - 1)/4\delta. \tag{6.18}$$

7. Singularities

The streamline pattern shown in figure 2 occurs while the thickness of the shear layer is growing at a rate that is independent of x. According to inviscid theory, if $k(t) = \partial^2 p/\partial x^2$ is such that $\beta(t)$, the solution to equation (5.12), has a zero when $t = t_c$ then as $t \to t_c$ the thickness of the layer grows without bound while at any fixed *finite* y

$$v \to y/(t_c - t) \quad \text{and} \quad u \to B'(0)y - (x - x_R(t_c))/(t_c - t). \tag{7.1}$$

Thus, the velocity components become unbounded at $t = t_c$. For example, when $k \equiv$ constant $= \sigma^2, \beta = \cos(\sigma t)$ and $t_c = \pi/2\sigma$.

Viscosity alone cannot always prevent the flow from developing singularities. For

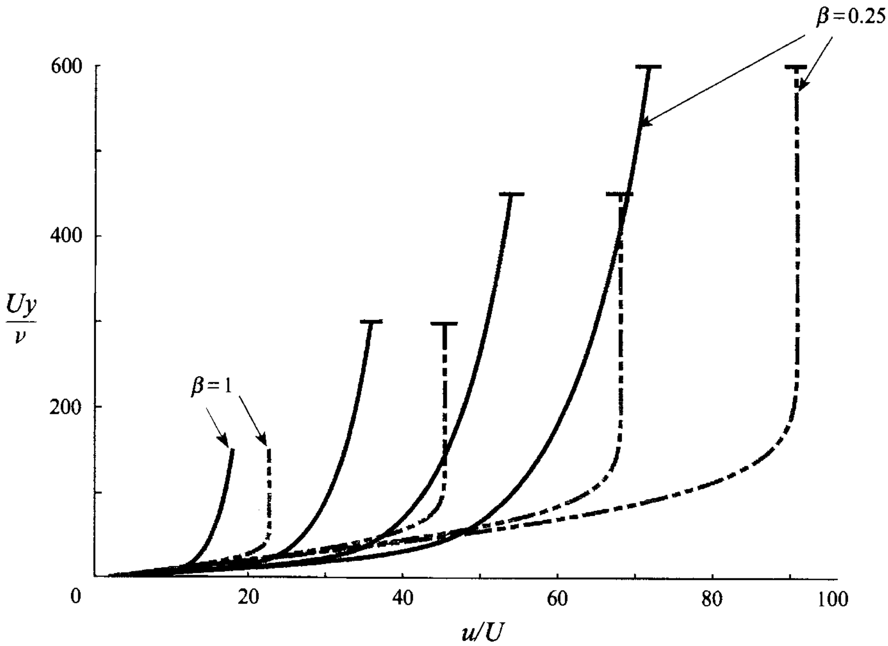

FIGURE 5. Changes in typical shear profiles as β varies in time.

example, the corresponding solution to the Navier–Stokes equations is

$$v = -\beta'(t)y/\beta(t) \quad \text{and} \quad u = (C(\tau, Y) - \beta'(t)(x - x_R(t)))/\beta(t), \tag{7.2}$$

where $C(\tau, Y)$ is given by (6.9). This solution also develops singularities if $\beta(t_c) = 0$. In fact, as $t \to t_c$ relations (7.1) continue to hold with the constant $B'(0)$ replaced by the constant

$$(\pi\nu\tau_c)^{-1/2}\int_0^\infty B'(Y)\exp(-Y^2/4\nu\tau_c)\mathrm{d}Y, \quad \text{where} \quad \tau_c = \int_0^{t_c}[\beta(t)]^2\mathrm{d}t. \tag{7.3}$$

The fact is, *it is not viscosity alone that prevents the formation of singularities, but viscosity together with correctly specified boundary, or auxiliary, conditions.* Both the inviscid and viscous solutions develop singularities if $k(t)$ is such that $\beta(t)$ has a zero. In practice, $\beta(t)$ is determined by the global flow, not just by local conditions near the plane $x = 0$. When a correct description of the interaction between the flow in the shear layer and the outer mainstream can be found, the outer flow always adjusts so that $\beta(t)$ has no zeros. Then, neither the viscous solution nor the inviscid solution develop singularities. Of course, the Euler equations governing unsteady plane flows have many solutions that develop singularities in a finite time. Those that describe irrotational flows, or flows with constant vorticity, are also exact solutions of the Navier–Stokes equations: it is not viscosity but boundary, or other auxiliary, conditions that prevent such flows from occurring. Figure 5 depicts the changes in two typical shear profiles at the cross-sections $x = x_R(t)$ when β varies in time. As β decreases, the thickness of the layer and the magnitude of u at any particle grow like β^{-1}. For the example represented by the broken curves, u is given by (7.2) with $C(\tau, Y)$ given by (3.11); for the other example u is given by (6.11).

8. Axis of rotation varying in time

So far the direction in which the flow variables are changing rapidly with distance has been fixed relative to an inertial frame. In this section we state the modifications that should be made to the analysis when this direction rotates. The representations are then used to describe flows in the vicinity of a pitching channel.

The reference axes, which rotate with a time-varying angular velocity

$$\boldsymbol{\Omega}(t) = (\Omega_1(t), \Omega_2(t), \Omega(t)) \tag{8.1}$$

relative to an inertial frame, are chosen so that the x_3-axis again coincides with the direction in which the flow variables are changing rapidly with distance y. Then, the Navier–Stokes equations (2.1) still have exact solutions of the form given by (2.5) and (2.6); $\mathbf{a}(t,y)$ and $v(t,y)$ still satisfy (2.10), but (2.11) for the $b_i(t,y)$ must be replaced by the equations

$$\frac{\partial b_i}{\partial t} + v\frac{\partial b_i}{\partial y} + (a_{ij} + 2\Omega J_{ij})b_j - J_{ij}(2\Omega_j v + \Omega_j' y) + p_i = \nu\frac{\partial^2 b_i}{\partial y^2}. \tag{8.2}$$

The $p_i(t,y)$ are determined from the conditions

$$\frac{\partial p_i}{\partial y} = -(2J_{jk}\Omega_k a_{ji} + J_{ij}\Omega'), \tag{8.3}$$

and $p_0(t,y)$ from the condition

$$\frac{\partial p_0}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial t} - v\frac{\partial v}{\partial y} + 2(\Omega_2 b_1 - \Omega_1 b_2). \tag{8.4}$$

8.1. *Flow referred to fixed axes*

If $\bar{\mathbf{x}}$ denote the Cartesian coordinates of a point referred to axes whose origin coincides with that of the rotating axes but which are *not rotating* relative to inertial axes, we can write

$$\mathbf{x} = \mathbf{r}(t)\bar{\mathbf{x}} \quad \text{where} \quad x_3 = y. \tag{8.5}$$

The elements of the rotation matrix $\mathbf{r}$ can be written

$$r_{ij} = \cos\theta\delta_{ij} + (1-\cos\theta)\Omega_i\Omega_j/(\theta')^2 + (\epsilon_{ijk}\Omega_k/\theta')\sin\theta, \tag{8.6}$$

where

$$\Omega_3 = \Omega(t) \quad \text{and} \quad \theta'(t) = (\Omega_i(t)\Omega_i(t))^{1/2}. \tag{8.7}$$

The fluid velocity relative to the non-rotating axes, $\bar{\mathbf{u}}$, can be written in terms of the velocity, $\mathbf{u}$, relative to the rotating axes, as

$$\bar{\mathbf{u}} = \mathbf{r}^T\mathbf{u} + \boldsymbol{\Omega}\times\bar{\mathbf{x}} \quad \text{where} \quad u_3 = v. \tag{8.8}$$

In terms of the *similarity* variable

$$y = \mathbf{n}(t)\cdot\bar{\mathbf{x}}, \quad \text{where} \quad n_i(t) = r_{3i}(t), \quad i = 1,2,3, \tag{8.9}$$

$$\bar{\mathbf{u}} = (\bar{\mathbf{a}}(t,y) - 2\mathbf{n}(t)\mathbf{n}'(t))\bar{\mathbf{x}} + \bar{\mathbf{b}}(t,y). \tag{8.10}$$

The pressure $\bar{p}$ is determined from the relation

$$\bar{p}/\rho + \mathbf{f}(t)\cdot\bar{\mathbf{x}} = (\bar{\mathbf{k}}(t) + 4\mathbf{n}'(t)\mathbf{n}'(t))\bar{\mathbf{x}}\bar{\mathbf{x}}/2 + \mathbf{P}(t,y)\cdot\bar{\mathbf{x}} + Q(t,y); \tag{8.11}$$

$\boldsymbol{f}(t)$ is the acceleration of the reference axes relative to an inertial frame and $\boldsymbol{n}(t)$ is a unit vector. The $\bar{\mathbf{a}}(t, y)$ and $v(t, y)$ satisfy the equations

$$\frac{\partial v}{\partial y} + \operatorname{tr}(\bar{\mathbf{a}}) = 0 \tag{8.12}$$

and

$$\frac{\partial \bar{\mathbf{a}}}{\partial t} + v\frac{\partial \bar{\mathbf{a}}}{\partial y} + \bar{\mathbf{a}}^2 + \bar{\boldsymbol{k}}(t) = \nu \frac{\partial^2 \bar{\mathbf{a}}}{\partial y^2}. \tag{8.13}$$

In addition,

$$\bar{\mathbf{a}}\boldsymbol{n} = \boldsymbol{n}\bar{\mathbf{a}} = \boldsymbol{n}'; \tag{8.14}$$

$\bar{\boldsymbol{k}}(t)$ satisfies the constraints

$$\bar{\boldsymbol{k}}^T = \bar{\boldsymbol{k}} \quad \text{and} \quad \bar{\boldsymbol{k}}\boldsymbol{n} + \boldsymbol{n}'' = \mathbf{0}. \tag{8.15}$$

Equations (8.12)–(8.15) are compatible.

Once $v(t, y)$ and the $\bar{\mathbf{a}}(t, y)$ have been determined from the above equations and the associated auxiliary conditions, $\bar{\boldsymbol{b}}(t, y)$ is determined from the equation

$$\frac{\partial \bar{\boldsymbol{b}}}{\partial t} + v\frac{\partial \bar{\boldsymbol{b}}}{\partial y} + \bar{\mathbf{a}}\bar{\boldsymbol{b}} + R\boldsymbol{n} + \boldsymbol{P} = \nu\frac{\partial^2 \bar{\boldsymbol{b}}}{\partial y^2}, \tag{8.16}$$

where

$$R = \nu\frac{\partial^2 v}{\partial y^2} - v\frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} \tag{8.17}$$

and

$$\frac{\partial \boldsymbol{P}}{\partial y} = 2(\boldsymbol{n}'' + \bar{\mathbf{a}}\boldsymbol{n}') \quad \text{with} \quad \boldsymbol{n}\cdot\boldsymbol{P} = 0. \tag{8.18}$$

Also, $Q(t, y)$ is determined from the condition that

$$\frac{\partial Q}{\partial y} = R + 2\boldsymbol{n}'\cdot\bar{\boldsymbol{b}}. \tag{8.19}$$

8.2. *Generalized Kelvin flows*

In the special case when $\bar{\mathbf{a}} = \bar{\mathbf{a}}(t)$,

$$\left.\begin{aligned} &v = v_0(t) + v_1(t)y, \quad \text{where} \quad v_1 = -\operatorname{tr}(\bar{\mathbf{a}}), \quad R = -(v_0' + v_1 v_0) - (v_1' + v_1^2)y, \\ \text{and} \quad &\boldsymbol{P} = 2(\boldsymbol{n}'' + \bar{a}^T\boldsymbol{n}')y + \boldsymbol{P}_0(t) \quad \text{with} \quad \boldsymbol{n}\cdot\boldsymbol{P}_0 = 0. \end{aligned}\right\} \tag{8.20}$$

Also,

$$\bar{\boldsymbol{b}} = v\boldsymbol{n} - 2y\boldsymbol{n}' + \bar{\boldsymbol{b}}_0(t) + \bar{\boldsymbol{b}}_1(t)y + [\boldsymbol{l}(t)]^{-1}\boldsymbol{C}(\tau, Y) \tag{8.21}$$

$\bar{\boldsymbol{b}}_1$ and $\bar{\boldsymbol{b}}_0$ satisfy the equations

$$\bar{\boldsymbol{b}}_1' + v_1\bar{\boldsymbol{b}}_1 + \bar{\mathbf{a}}\bar{\boldsymbol{b}}_1 = 2(\bar{\mathbf{a}} - \bar{\mathbf{a}}^T)\boldsymbol{n}' \quad \text{with} \quad \boldsymbol{n}\cdot\bar{\boldsymbol{b}}_1 = 0, \tag{8.22}$$

and

$$\bar{\boldsymbol{b}}_0' + \bar{\mathbf{a}}\bar{\boldsymbol{b}}_0 + \boldsymbol{P}_0 + v_0\bar{\boldsymbol{b}}_1 = 0 \quad \text{with} \quad \boldsymbol{n}\cdot\bar{\boldsymbol{b}}_0 = 0. \tag{8.23}$$

The 3×3 matrix $\boldsymbol{l}(t)$ satisfies the (compatible) equations

$$\boldsymbol{l}'' + \boldsymbol{l}\bar{\boldsymbol{k}} = 0, \quad \boldsymbol{l}'\boldsymbol{n} = \boldsymbol{l}\boldsymbol{n}' \quad \text{and} \quad N\boldsymbol{l} = \boldsymbol{n}, \tag{8.24}$$

where $\boldsymbol{N}$ is a constant vector. In terms of $\boldsymbol{l}$,

$$\bar{\mathsf{a}} = \boldsymbol{l}^{-1}\boldsymbol{l}'. \tag{8.25}$$

Also, $\tau(t)$ is given by (6.2), and $Y = \beta(t)(y - y_0(t))$ where

$$\beta(t) = \exp\left(-\int_0^t v_1(t')\mathrm{d}t'\right). \tag{8.26}$$

Each of the three components of $\boldsymbol{C}(\tau, Y)$ satisfies the diffusion equation together with the (compatible) constraint

$$\boldsymbol{N}\cdot\boldsymbol{C} = \boldsymbol{0}. \tag{8.27}$$

9. Flows in a pitching channel

The representations obtained in §8 may be used to describe two-dimensional Couette flows that occur in a channel that pitches about an inertial axis normal to the plane of flow. The $\bar{x}_3$-axis is taken as the axis of rotation. The boundaries of the channel are parallel to the line ($y = 0$) whose equation is $\bar{x}_2 = \tan(\theta)\bar{x}_1$; $\theta(t)$ is arbitrary. Then,

$$\boldsymbol{n} = (-\sin\theta, \cos\theta, 0), \quad \boldsymbol{n}' = -\gamma(\cos\theta, \sin\theta, 0) = -\gamma\boldsymbol{t}, \quad \text{where} \quad \gamma = \theta', \tag{9.1}$$

while

$$\bar{x}_1 = x\cos\theta - y\sin\theta \quad \text{and} \quad \bar{x}_2 = x\sin\theta + y\cos\theta. \tag{9.2}$$

The equations of the channel walls are given by (9.2) with $y = y_0$ and $y = y_1$.

When $\bar{u}_3 \equiv 0$, conditions (9.1), together with the restictions (8.14), imply that

$$\bar{\mathsf{a}} = \boldsymbol{n}\boldsymbol{n}' + \boldsymbol{n}'\boldsymbol{n} + \boldsymbol{n}'\boldsymbol{n}'/\gamma^2. \tag{9.3}$$

According to (8.12)–(8.15) and (9.1), $v(t, y)$ and $a(t, y)$ satisfy the equations

$$\frac{\partial v}{\partial y} + a = 0 \quad \text{and} \quad \frac{\mathrm{d}a}{\mathrm{d}t} + a^2 + k = \nu\frac{\partial^2 a}{\partial y^2}, \quad \text{where} \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial y}; \tag{9.4}$$

$k(t)$ is arbitrary. Also, equations (8.16)–(8.19) imply that

$$\bar{\boldsymbol{b}} = v\boldsymbol{n} + b\boldsymbol{t}, \tag{9.5}$$

where $b(t, y)$ satisfies the equation

$$\frac{\mathrm{d}b}{\mathrm{d}t} + ab + P_0 - 2\gamma' y = \nu\frac{\partial^2 b}{\partial y^2}; \tag{9.6}$$

$P_0(t)$ is arbitrary. According to (8.10), (9.1) and (9.5),

$$\bar{\boldsymbol{u}} = [v(t, y) + \gamma x]\boldsymbol{n} + [a(t, y)x + b(t, y) - \gamma y]\boldsymbol{t}, \quad \text{where} \quad x = \boldsymbol{t}\cdot\boldsymbol{x} \quad \text{and} \quad y = \boldsymbol{n}\cdot\boldsymbol{x}. \tag{9.7}$$

Also, the pressure $\bar{p}$ is determined from the relation

$$\bar{p}/\rho + \boldsymbol{f}\cdot\boldsymbol{x} = kx^2/2 + \gamma^2(x^2 + y^2)/2 - \gamma' xy + (2\gamma v + P_0)x + Q(t, y), \tag{9.8}$$

where

$$\frac{\partial Q}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2} - v \frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} - 2\gamma b. \tag{9.9}$$

When $\gamma \equiv 0$ and $b \equiv 0$, equations (9.1)–(9.9) describe the Blasius–Hiemenz (1908, 1911) flow at a forward stagnation point.

For the flow in a pitching channel $a \equiv 0$, $v \equiv 0$ and $b(t, y)$ satisfies the equation

$$\frac{\partial b}{\partial t} + P_0(t) - 2\gamma'(t)y = \nu \frac{\partial^2 b}{\partial y^2}. \tag{9.10}$$

The no-slip condition at the channel boundaries requires that

$$b(t, y_0) = b(t, y_1) = 0. \tag{9.11}$$

We suppose that $P_0 \equiv$constant. Also, prior to $t = 0$ the flow in the non-pitching channel is a steady Couette flow with

$$b = -(P_0/2\nu)(y - y_0)(y_1 - y). \tag{9.12}$$

For $t > 0$ we write

$$b = -(P_0/2\nu)(y - y_0)(y_1 - y) + 2\gamma(t)y + c(t, y), \tag{9.13}$$

where $c(t, y)$ satisfies the diffusion equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial y^2}, \tag{9.14}$$

with

$$c(0, y) = 0 \quad \text{for} \quad y_0 < y < y_1, \quad \text{and} \quad c(t, y) = -2\gamma(t)y \quad \text{at} \quad y = y_0 \quad \text{and} \quad y = y_1. \tag{9.15}$$

If t is measured in units of $(y_1 - y_0)^2/\nu$, the solution to (9.14) satisfying conditions (9.15) can be written as

$$c(t, y) = -2 \int_0^t [y_0 k(t - t', z_1) + y_1 k(t - t', z)]\gamma(t')\mathrm{d}t', \tag{9.16}$$

where

$$z_1 = (y_1 - y)/(y_1 - y_0) \quad \text{and} \quad z = (y - y_0)/(y_1 - y_0). \tag{9.17}$$

$k(t, z)$, whose Laplace transform with respect to t is $\sinh(zs^{\frac{1}{2}})/\sinh(s^{\frac{1}{2}})$, can be represented by the series

$$k(t,z) = (4\pi t^3)^{-1/2} \sum_{n=0}^{\infty} (1-z+2n)\exp(-(1-z+2n)^2/4t) - (1+z+2n)\exp(-(1+z+2n)^2/4t). \tag{9.18}$$

When γ changes discontinuously at $t = 0$ and thereafter remains constant, the representation for b is best rewritten as

$$b = -(P_0/2\nu)(y - y_0)(y_1 - y) + 2\gamma(t)(y_0\bar{b}_0(t, z) + (y_1 - y_0)\bar{b}_1(t, z)), \tag{9.19}$$

where

$$\bar{b}_0(t, z) = 4\pi^{-1} \sum_{n=1}^{\infty} (2n - 1)^{-1} \exp(-(2n - 1)^2\pi^2 t)) \sin((2n - 1)\pi z) \tag{9.20}$$

and

$$\bar{b}_1(t, z) = 2\pi^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} \exp(-n^2\pi^2 t)) \sin(n\pi z). \tag{9.21}$$

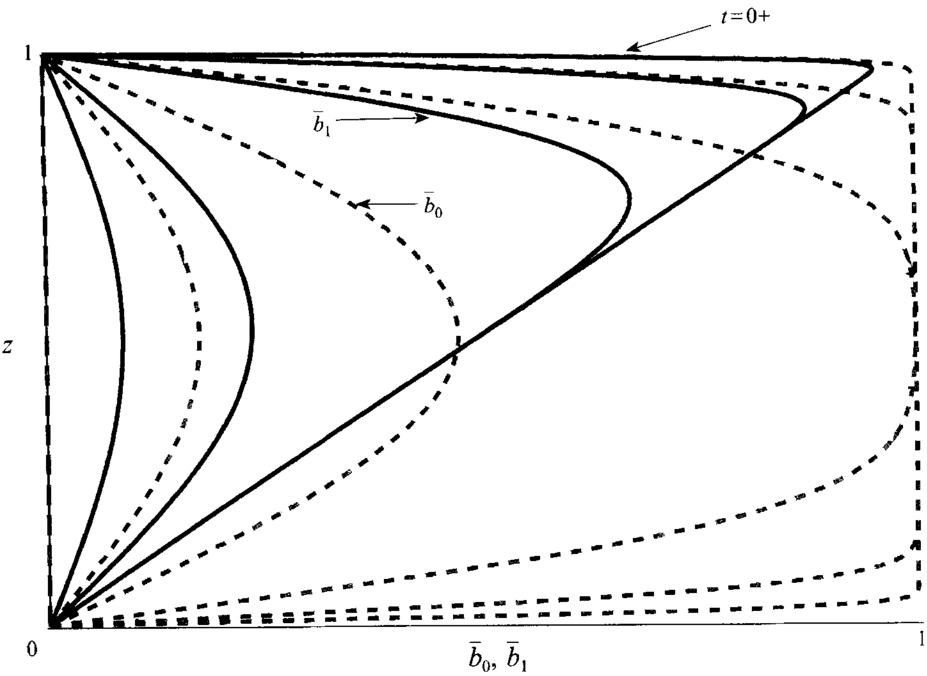

FIGURE 6. The profiles of $\bar{b}_0$ and $\bar{b}_1$ at various values of t.

Figure 6 depicts the profiles of $\bar{b}_0(t,z)$ and $\bar{b}_1(t,z)$ with increasing t. As $t \rightarrow \infty$, $(\bar{b}_0, \bar{b}_1) \rightarrow 0$.

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